

Using GENERATINGFUNCTIONOLOGY to Enumerate Distinct-Multiplicity Partitions

Doron ZEILBERGER¹*In fond memory of Guru Herbert Saul WILF (28 Sivan 5691- 12 Tevet 5772) zecher gaon l'bracha***Preamble**

About a year ago, Herb Wilf[W1] posed, on-line, eight intriguing problems. I don't know the *answer* to any of them, but I will say something about the sixth question.

Herb Wilf 6th Question: *Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$. See Sloane's sequence **A098859** for a table of values. Find any interesting theorems about $f(n)$...*

First, I will explain how to compute the first few terms of $f(n)$. Shalosh can easily get the first 250 terms, but as n gets larger it gets harder and harder to compute, unlike its unrestricted cousin, $p(n)$. I conjecture that the fastest algorithm takes exponential time, but I have no idea how to prove that claim. I am impressed that, according to Sloane, Maciej Ireneusz Wilczynski computed 508 terms.

Recall that the generating function for the number of integer partitions of n whose largest part is $\leq m$, $p_m(n)$, is the very simple rational function

$$\sum_{n=0}^{\infty} p_m(n)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \quad .$$

The main purpose of this note is to describe, using *Generatingfunctionology*, so vividly and lucidly *preached* in W's **classic book** [W2], how to compute the generating function (that also turns out to be rational) for the number of partitions of n whose largest part is $\leq m$ **and** all its (nonzero) multiplicities are distinct, let's call it $f_m(n)$. As m gets larger, the formulas get more and more complicated, but we sure do have an **answer**, in the sense of the **classic article** [W3], for any *fixed* m , but of course not for a *symbolic* m .

Even more is true! Because, like $\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$, the generating function of $f_m(n)$, $\sum_{n=0}^{\infty} f_m(n)q^n$, turns out (as we will see) to only have roots-of-unity poles, whose highest order is m , it follows that $f_m(n)$ is a *quasi-polynomial* of degree $m-1$ in n . Now that's a very good answer! (in W's sense, albeit only for a fixed m).

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How to Compute Many terms of $f(n)$?

$p_m(n)$ is very easy to compute. For example, one may use the recurrence

$$p_m(n) = p_{m-1}(n) + \sum_{i=1}^{\lfloor n/m \rfloor} p_{m-1}(n - mi) \quad ,$$

together with the *initial condition* $p_1(n) = 1$, $p_m(0) = 1$.

How can we adapt this in order to compute $f_m(n)$? The contribution from the partitions counted by $f_m(n)$ where m does not show up is $f_{m-1}(n)$, in analogy with the $p_{m-1}(n)$ term in the above recurrence. But if m *does* show up, it does so with a certain multiplicity, i , say, where $1 \leq i \leq \lfloor n/m \rfloor$, and removing these i copies of m results in a partition counted by $f_{m-1}(n - mi)$ -so all its multiplicities are different- and **in addition** none of these multiplicities may be i . Continuing, we are forced to introduce a much more general discrete function $f_m(n; S)$ whose arguments are m and n and a set of “forbidden multiplicities”, S .

So let's define $f_m(n; S)$ to be the number of partitions of n with parts $\leq m$, with all its multiplicities distinct **and** none of these multiplicities belonging to S . Our intermediate object of desire, $f_m(n)$, is simply $f_m(n; \emptyset)$, and the ultimate object, $f(n)$, is $f_n(n; \emptyset)$.

The recurrence for $f_m(n; S)$ is, naturally

$$f_m(n; S) = f_{m-1}(n; S) + \sum_{i=1, i \notin S}^{\lfloor n/m \rfloor} f_{m-1}(n - im; S \cup \{i\}) \quad ,$$

because once we decided on the number of times m shows up, let's call it i , where i is between 1 and $\lfloor n/m \rfloor$ and $i \notin S$, the partition (of $n - im$) obtained by removing these i copies of m must forbid the set of multiplicities $S \cup \{i\}$.

In the Maple package DMP, procedure **qnmS(n,m,S)** implements $f_m(n; S)$ and procedure **qn(n)** implements $f(n)$.

Inclusion-Exclusion

Let $\mathcal{P}_m(n)$ be the set of partitions of n whose parts are all $\leq m$, in other words, the set that $p_m(n)$ is counting. Consider the set of *all* partitions whose largest part is $\leq m$, where we write a partition in *frequency notation*:

$$\mathcal{P}_m := \{1^{a_1} 2^{a_2} \dots m^{a_m} \mid a_1, \dots, a_m \geq 0\} \quad .$$

For example $1^3 2^5 4^2$ is the partition of twenty-one usually written as 4422222111. Introducing *symbols* x_1, x_2, \dots, x_m , we define the *Weight* of a partition to be

$$\text{Weight}(1^{a_1} 2^{a_2} \dots m^{a_m}) := x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} \quad .$$

The *weight-enumerator* of \mathcal{P}_m is, by **ordinary**-generatingfunctionology

$$Weight(\mathcal{P}_m) = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_m)} \quad ,$$

since we make m *independent* decisions:

- how many copies of 1? (Weight enumerator = $1 + x_1 + x_1^2 + \dots = (1-x_1)^{-1}$) ,
- how many copies of 2? (Weight enumerator = $1 + x_2 + x_2^2 + \dots = (1-x_2)^{-1}$) ,
- ...
- how many copies of m ? (Weight enumerator = $1 + x_m + x_m^2 + \dots = (1-x_m)^{-1}$).

But we want to find the weight-enumerator of the much-harder-to-weight-count set

$$\mathcal{F}_m := \{1^{a_1} 2^{a_2} \dots m^{a_m} \mid a_1, \dots, a_m \geq 0; a_i \neq a_j \text{ (if } a_i > 0, a_j > 0)\} \quad .$$

Calling the members of \mathcal{F}_m *good*, we see that a member of \mathcal{P}_m is good if it does **not** belong to any of the following $\binom{m}{2}$ sets, S_{ij} $1 \leq i < j \leq m$:

$$S_{ij} := \{1^{a_1} 2^{a_2} \dots m^{a_m} \in \mathcal{P}_m \mid a_i = a_j > 0\} \quad .$$

By inclusion-exclusion, the weight-enumerator of \mathcal{F}_m is

$$\sum_G (-1)^{|G|} Weight \left(\bigcap_{i,j \in G} S_{ij} \right) \quad ,$$

where the summation ranges over **all** $2^{m(m-1)/2}$ subsets of $\{(i,j) \mid 1 \leq i < j \leq m\}$.

But the G 's can be naturally viewed as *labeled graphs* on m vertices. Such a graph has several connected components, and together they naturally induce a *set partition* $\{C_1, C_2, \dots, C_r\}$ of $\{1, 2, \dots, m\}$. We have:

$$Weight \left(\bigcap_{i,j \in G} S_{ij} \right) = \prod_{i=1}^r weight(C_i) \quad ,$$

where if $|S| = 1$, $S = \{s\}$, say, then $weight(S) = \frac{1}{1-x_s}$, and if $|S| = d > 1$, $S = \{s_1, s_2, \dots, s_d\}$, say, then

$$weight(S) = \frac{x_{s_1} x_{s_2} \cdots x_{s_d}}{1 - x_{s_1} x_{s_2} \cdots x_{s_d}} \quad .$$

To justify the latter, note that if vertices s_1, s_2, \dots, s_d all belong to the same connected component of our graph then, by *transitivity*, we have that all $a_{s_1} = a_{s_2} = \dots = a_{s_d} > 0$, and the weight-enumerator is the infinite geometric series

$$\sum_{\alpha=1}^{\infty} (x_{s_1} \cdots x_{s_d})^\alpha = \frac{x_{s_1} x_{s_2} \cdots x_{s_d}}{1 - x_{s_1} x_{s_2} \cdots x_{s_d}} \quad .$$

But quite a few graphs correspond to any one set-partition. To find out the coefficients in front, for any set-partition $\{C_1, C_2, \dots, C_r\}$ of $\{1, \dots, m\}$ we must find

$$\sum_G (-1)^{|G|} \quad ,$$

summed over all the graphs that gives rise to the above set partition. But this is the product of the analogous sums where one focuses on one connected component at a time, and then multiplies everything together.

Let's digress and figure out $\sum_G (-1)^{|G|}$ over all *connected labeled graphs* on n vertices. For the sake of clarity, let's, more generally, figure out $\sum_G y^{|G|}$ with a general variable y .

By **exponential-generatingfunctionology**[W2] (see also [Z]), this sum is nothing but the coefficient of $t^n/n!$ in

$$\log \left(\sum_{i=0}^{\infty} (1+y)^{\binom{i}{2}} \frac{t^i}{i!} \right) \quad .$$

Going back to $y = -1$, we see that we need the coefficient of $t^n/n!$ in

$$\begin{aligned} \log \left(\sum_{i=0}^{\infty} (1+(-1))^{\binom{i}{2}} \frac{t^i}{i!} \right) &= \log \left(\sum_{i=0}^{\infty} 0^{\binom{i}{2}} \frac{t^i}{i!} \right) = \log(1+t) \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \frac{t^n}{n!} \quad . \end{aligned}$$

So the desired sum is $(-1)^{n-1} (n-1)!$.

Let's define for any set of positive integers, S ,

$$\text{mishkal}(S) = \begin{cases} 1/(1-x_s), & \text{if } |S| = 1 \text{ where } S = \{s\} ; \\ (-1)^{d-1} (d-1)! (x_{s_1} \cdots x_{s_d}) / (1-x_{s_1} \cdots x_{s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1, \dots, s_d\}. \end{cases}$$

For any set partition $C = \{C_1, \dots, C_r\}$ let's define

$$\text{Mishkal}(C) = \text{mishkal}(C_1) \cdots \text{mishkal}(C_r) \quad .$$

It follows that the weight-enumerator of \mathcal{F}_m according to $\text{Weight}(1^{a_1} 2^{a_2} \dots m^{a_m}) := x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ is

$$\sum_C \text{Mishkal}(C) \quad ,$$

where the sum has B_m terms (B_m being the Bell numbers), one for each set-partition of $\{1, \dots, m\}$.

Finally, to get an “explicit” formula (as a sum of B_m terms, each a simple rational function of q), for the generating function $\sum_{n=0}^{\infty} f_m(n) q^n$, all we need is replace x_i by q^i , for $i = 1 \dots m$, getting

$$\sum_{n=0}^{\infty} f_m(n) q^n = \sum_C \text{Poids}(C) \quad ,$$

where for a set partition $C = \{C_1, \dots, C_r\}$

$$Poids(C) = poids(C_1) \cdots poids(C_r) \quad ,$$

and where for an individual set S :

$$poids(S) = \begin{cases} 1/(1 - q^s), & \text{if } |S| = 1 \text{ where } S = \{s\} ; \\ (-1)^{d-1} (d-1)! q^{s_1 + \dots + s_d} / (1 - q^{s_1 + \dots + s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1, \dots, s_d\}. \end{cases} \quad .$$

It follows that indeed $f_m(n)$ is a quasi-polynomial of degree $m - 1$ in n . Furthermore, since the only pole that has multiplicity m is $q = 1$, it follows that the leading term (of degree $m - 1$) is a pure polynomial.

The generating function, $\sum_{n=0}^{\infty} f_m(n) q^n$, for any desired positive integer m , is implemented in procedure `GFmq(m,q)` in the Maple package `DMP`. For the Weight-enumerator (or rather with x_i replaced by $q^i x_i$, for $i = 1, \dots, m$), see `GFmxq(m,x,q)`. Since the Bell numbers grow very fast, the formulas get complicated rather fast, but *in principle* we do have a very nice *answer* for any specific m , but *in practice*, for large m it is only “nice” in principle. Of course it is *anything but* nice when viewed also as function of m , and that’s why $f(n) = f_n(n)$ is **probably** very hard to compute for larger n .

To see the outputs of `GFmq(m,q)` for $1 \leq m \leq 8$ see:

<http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP3> .

Asymptotics

Recall that Hardy and Ramanujan tell us that as n goes to infinity, $p(n)$ is asymptotic to $\frac{1}{4n\sqrt{3}} \exp(C\sqrt{n})$ where $C = \sqrt{2/3}\pi = 2.565099661\dots$, and hence $\log p(n)/\sqrt{n}$ converges to C . By looking at the sequence $\log f(n)/\sqrt{n}$ for $1 \leq n \leq 508$, it seems that this too converges to a limit, that appears to be a bit larger than 1.517 (but of course way less than 2.565099661...). Let’s call that constant the *Wilf constant*.

The numerical evidence is here: <http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP4> .

Let me conclude with two challenges.

- Prove that the Wilf constant exists.
- Determine the exact value of the Wilf constant (if it exists) in terms of π or other famous constants. Failing this, find non-trivial rigorous lower and upper bounds.

References

- [W1] Herbert Wilf, *Some Unsolved problems*,
<http://www.math.upenn.edu/~Ewilf/website/UnsolvedProblems.pdf>, posted: Dec. 13, 2010.
(viewed Jan. 16, 2012)
- [W2] Herbert S. Wilf, **“Generatingfunctionology”**, Academic Press, First edition 1990, Second Edition 1994. Third Edition: AK Peters, 2005. Second edition is freely downloadable from
<http://www.math.upenn.edu/~Ewilf/gfology2.pdf> .
- [W3] Herbert S. Wilf, *What is an Answer?*, Amer. Math. Monthly **89** (1982), 289-292.
- [Z] Doron Zeilberger, *Enumerative and Algebraic Combinatorics*, in: **“Princeton Companion to Mathematics”** , (Timothy Gowers, ed.), Princeton University Press, 550-561. Available from:
<http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/enu.pdf>